06 Moments of a sample

The mean of a sample is its characteristic of order 1. Characteristics of higher orders are defined by formulas which are analogous to that yielding the mean.

Given a natural r and a sample $y = (y_1, y_2, \ldots, y_N)$, its r-th **moment**, or **moment of order** r (about a number a) is

$$\mu_{r,a}(y) := \frac{1}{N}\cdot \sum_{j=1}^{N} (y_j - a)^r .$$

We simply write $\mu_{r,a}$, if it is clear we deal with y with. We do not indicate $a = 0$, i.e., $\mu_r := \mu_{r,0}$, and we refer to this quantity as a r-th (**raw**) **moment**.

Notice that the first (raw) moment ($a = 0$, $r = 1$),

$$\mu_1(y) := \frac{1}{N}\cdot \sum_{j=1}^{N} y_j ,$$

is the (arithmetic) **average** of numbers $y_1, y_2, \ldots, y_N$, is the mean(y), also denoted by $\overline{y}$. The mean plays an inevitable role in the next definition.

The r-th moment around the mean is referred to as a r-th **central moment**, and denoted by M_r. So, a r-th central moment of the sample $y = (y_1, y_2, \ldots, y_N)$ is

$$\gamma_r(y) := \frac{1}{N}\cdot \sum_{j=1}^{N} (y_j - \overline{y})^r ,$$

where, as usually, $N = \text{size}(y)$.

Obviously, above formulas can be adapted to multences: raw moments and central moments of the multence (x, m) are equal to that of the sequence this multence is generated by. This results with following formulas:

$$\mu_{r,a}(x, m) := \frac{1}{N} \sum_{j=1}^{n} m_j \cdot (x_j - a)^r , \ \mu_r(x, m) := \frac{1}{N} \sum_{j=1}^{n} m_j \cdot x_j^{\,r}$$

$$\gamma_r(x, m) = \mu_{r,\xi}(x, m) := \frac{1}{N} \sum_{j=1}^{n} m_j \cdot (x_j - \xi)^r ,$$

where ξ denotes the mean of the multence (x, m), i.e.

$$\xi := \mu_1(x, m) := \frac{1}{N} \sum_{j=1}^{n} m_j \cdot x_j .$$

Usually we omit the indications to a sample, e.g., instead of $\mu_r(y)$ we write μ_r.

A convenient way to produce these quantities consists in using multence tables.

Example–4. Still working with data y taken from the firm *We20*, Example-1, we calculate, for example, the second central moment of y. We perform these calculation in the multence table enlarged by columns storing differences $x_j - \xi$, where $\gamma = \text{mean}\,(x, m) = \text{mean}(y)$, their squares, $(x_j - \xi)^2$, and the products $m_j{\cdot}(x_j - \xi)^2$.

The initial table, already enlarged by mentioned columns (and by the lower row, identified by Σ, to be stored by needed sums of elements listed in columns):

j	x_j	m_j	$m_j{\cdot}x_j$	$x_j - \xi$	$(x_j - \xi)^2$	$m_j{\cdot}(x_j - \xi)^2$
1	2	2				
2	2.1	1				
3	2.2	2				
4	2.9	4				
5	3.1	1				
6	3.3	3				
7	3.5	2				
8	3.8	1				
9	4.3	1				
10	6.4	1				
11	7	1				
12	10	1				
Σ						

First we fill the column storing products $m_j{\cdot}x_j$, and we divide the sum of these products by $N = m_1 + m_2 + \ldots + m_n$ (the value produced by adding all values filling the column headed by m_j).

j	x_j	m_j	$m_j{\cdot}x_j$	$x_j - \xi$	$(x_j - \xi)^2$	$m_j{\cdot}(x_j - \xi)^2$
1	2	2	4			
2	2.1	1	2.1			
3	2.2	2	4.4			
4	2.9	4	11.6			
5	3.1	1	3.1			
6	3.3	3	9.9			
7	3.5	2	7			
8	3.8	1	3.8			
9	4.3	1	4.3			
10	6.4	1	6.4			
11	7	1	7			
12	10	1	10			
Σ		20	73.6			

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This way we obtain the mean, ξ, of the multence (*x*, *m*),

$$\xi = \mu_1 = \frac{73.6}{20} = 3.68.$$

Now we fill cells with appropriate differences, their squares and the products in columns. This way we get

j	x_j	m_j	$m_j \cdot x_j$	$x_j - \xi$	$(x_j - \xi)^2$	$m_j \cdot (x_j - \xi)^2$
1	2	2	4	–1.68	2.8224	5.6448
2	2.1	1	2.1	–1.58	2.4964	2.4964
3	2.2	2	4.4	–1.48	2.1904	4.3808
4	2.9	4	11.6	–0.78	0.6084	2.4336
5	3.1	1	3.1	–0.58	0.3364	0.3364
6	3.3	3	9.9	–0.38	0.1444	0.4332
7	3.5	2	7	–0.18	0.0324	0.0648
8	3.8	1	3.8	0.12	0.0144	0.0144
9	4.3	1	4.3	0.62	0.3844	0.3844
10	6.4	1	6.4	2.72	7.3984	7.3984
11	7	1	7	3.32	11.0224	11.0224
12	10	1	10	6.32	39.9424	39.9424
Σ		20	73.6			74.5520

Next we sum values filling the last column;

$$5.6448 + 2.4964 + 4.3808 + \ldots + 39.9424 = 74.5520,$$

and, finally, we find the required 2nd central moment

$$\gamma_2(x, m) = \frac{74.5520}{20} = 3.7276.$$

□ *Example–4*.

Example–5. Here we work with the condence produced in Example … This is the multence (*c*, *q*) = (3.0, 16; 5.0; 7.0, 2; 9.0). We calculate its central moments up to the 4th one.

We can start as in the previous example. First we fill columns headed with *ci* and *qi*, next we multiply $q_i \cdot c_i$ and sum figures staying in the column headed with *qi* and that in column headed with $q_i \cdot c_i$. This way we obtain the size of the sample,

$$N = q_1 + q_2 + q_3 + q_4 = 16 + 1 + 2 + 1 = 20,$$

and the first (raw) moment

$$q_1 \cdot c_1 + q_2 \cdot c_2 + q_3 \cdot c_3 + q_4 \cdot c_4 = 48 + 5 + 14 + 9 = 76.$$

In consequence, we find that the mean, $\xi = \frac{76}{20} = 3.8$.

Now we fill columns, with appropriate differences $(c_i - \xi)$ and their squares, and with products, next columns, next we sum figures filling columns headed with $q_i{\cdot}(c_j - \xi)^r$, $r = 2, 3, 4$. At last we divide these sums by the size N of the multence, and this way we get that quantities we looked for are

– the 2nd central moment $\gamma_2 = \frac{59.20}{20} = 2.96$,

– the 3rd central moment $\gamma_3 = \frac{199.68}{20} = 9.984$,

– the 4th central moment $\gamma_4 = \frac{949.503}{20} = 47.475$.

i	c_i	q_i	$q_i{\cdot}c_i$	$c_i - \xi$	$(c_j - \xi)^2$	$q_i{\cdot}(c_i - \xi)^2$	$q_i{\cdot}(c_j - \xi)^3$	$q_i{\cdot}(c_i - \xi)^4$
1	3.0	16	48	–0.8	0.64	10.24	–8.192	6.5536
2	5.0	1	5	1.2	1.44	1.44	1.728	2.0736
3	7.0	2	14	3.2	10.24	20.48	65.536	209.715
4	9.0	1	9	5.2	27.04	27.04	140.608	731.161
Σ		20	76			59.20	199.680	949.503

□ *Example–5.*

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